Chapter 3

Probability
Probability Experiments

Probability experiment
• An action, or trial, through which specific results (counts, measurements, or responses) are obtained.

Outcome
• The result of a single trial in a probability experiment.

Sample Space
• The set of all possible outcomes of a probability experiment.

Event
• Consists of one or more outcomes and is a subset of the sample space.
Probability Experiments

• **Probability experiment:** Roll a die

• **Outcome:** \( \{3\} \)

• **Sample space:** \( \{1, 2, 3, 4, 5, 6\} \)

• **Event:** \( \{\text{Die is even}\} = \{2, 4, 6\} \)
Example: Identifying the Sample Space

A probability experiment consists of tossing a coin and then rolling a six-sided die. Describe the sample space.

**Solution:**
There are two possible outcomes when tossing a coin: a head (H) or a tail (T). For each of these, there are six possible outcomes when rolling a die: 1, 2, 3, 4, 5, or 6. One way to list outcomes for actions occurring in a sequence is to use a tree diagram.
Solution: Identifying the Sample Space

Tree diagram:

The sample space has 12 outcomes:
{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}
Simple Events

Simple event

• An event that consists of a single outcome.
  ▪ e.g. “Tossing heads and rolling a 3”  \{H3\}

• An event that consists of more than one outcome is not a simple event.
  ▪ e.g. “Tossing heads and rolling an even number”  \{H2, H4, H6\}
Example: Identifying Simple Events

Determine whether the event is simple or not.

- You roll a six-sided die. Event B is rolling at least a 4.

Solution:
Not simple (event B has three outcomes: rolling a 4, a 5, or a 6)
Fundamental Counting Principle

Fundamental Counting Principle

• If one event can occur in \( m \) ways and a second event can occur in \( n \) ways, the number of ways the two events can occur in sequence is \( m \cdot n \).

• Can be extended for any number of events occurring in sequence.
Example: Fundamental Counting Principle

You are purchasing a new car. The possible manufacturers, car sizes, and colors are listed.

Manufacturer: Ford, GM, Honda
Car size: compact, midsize
Color: white (W), red (R), black (B), green (G)

How many different ways can you select one manufacturer, one car size, and one color? Use a tree diagram to check your result.
Solution: Fundamental Counting Principle

There are three choices of manufacturers, two car sizes, and four colors.

Using the Fundamental Counting Principle:

$$3 \cdot 2 \cdot 4 = 24 \text{ ways}$$
Types of Probability

Classical (theoretical) Probability

- Each outcome in a sample space is equally likely.

\[ P(E) = \frac{\text{Number of outcomes in event } E}{\text{Number of outcomes in sample space}} \]
Example: Finding Classical Probabilities

You roll a six-sided die. Find the probability of each event.

1. Event A: rolling a 3
2. Event B: rolling a 7
3. Event C: rolling a number less than 5

Solution:
Sample space: \{1, 2, 3, 4, 5, 6\}
Solution: Finding Classical Probabilities

1. Event A: rolling a 3  
   Event A = \{3\}  
   \[ P(\text{rolling a 3}) = \frac{1}{6} \approx 0.167 \]

2. Event B: rolling a 7  
   Event B = \{\} (7 is not in the sample space)  
   \[ P(\text{rolling a 7}) = \frac{0}{6} = 0 \]

3. Event C: rolling a number less than 5  
   Event C = \{1, 2, 3, 4\}  
   \[ P(\text{rolling a number less than 5}) = \frac{4}{6} \approx 0.667 \]
Types of Probability

Empirical (statistical) Probability

- Based on observations obtained from probability experiments.
- Relative frequency of an event.

\[ P(E) = \frac{\text{Frequency of event } E}{\text{Total frequency}} = \frac{f}{n} \]
Example: Finding Empirical Probabilities

1. A company is conducting a telephone survey of randomly selected individuals to get their overall impressions of the past decade (2000s). So far, 1504 people have been surveyed. What is the probability that the next person surveyed has a positive overall impression of the 2000s? (*Source: Princeton Survey Research Associates International*)

<table>
<thead>
<tr>
<th>Response</th>
<th>Number of times, $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive</td>
<td>406</td>
</tr>
<tr>
<td>Negative</td>
<td>752</td>
</tr>
<tr>
<td>Neither</td>
<td>316</td>
</tr>
<tr>
<td>Don’t know</td>
<td>30</td>
</tr>
</tbody>
</table>

$\Sigma f = 1504$
Solution: Finding Empirical Probabilities

<table>
<thead>
<tr>
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</tr>
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<td>752</td>
</tr>
<tr>
<td>Neither</td>
<td>316</td>
</tr>
<tr>
<td>Don’t know</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>$\Sigma f = 320$</td>
</tr>
</tbody>
</table>

$P(\text{positive}) = \frac{f}{n} = \frac{406}{1504} \approx 0.270$
Law of Large Numbers

- As an experiment is repeated over and over, the empirical probability of an event approaches the theoretical (actual) probability of the event.
Types of Probability

Subjective Probability

- Intuition, educated guesses, and estimates.
- e.g. A doctor may feel a patient has a 90% chance of a full recovery.
Example: Classifying Types of Probability

Classify the statement as an example of classical, empirical, or subjective probability.

1. The probability that you will get the flu this year is 0.1.

Solution:
Subjective probability (most likely an educated guess)
Example: Classifying Types of Probability

Classify the statement as an example of classical, empirical, or subjective probability.

2. The probability that a voter chosen at random will be younger than 35 years old is 0.3.

Solution:
Empirical probability (most likely based on a survey)
Example: Classifying Types of Probability

Classify the statement as an example of classical, empirical, or subjective probability.

3. The probability of winning a 1000-ticket raffle with one ticket is \( \frac{1}{1000} \).

Solution:
Classical probability (equally likely outcomes)
Range of probabilities rule

• The probability of an event $E$ is between 0 and 1, inclusive.
• $0 \leq P(E) \leq 1$
Complementary Events

Complement of event $E$

- The set of all outcomes in a sample space that are not included in event $E$.
- Denoted $E'$ (E prime)
- $P(E') + P(E) = 1$
- $P(E) = 1 - P(E')$
- $P(E') = 1 - P(E)$
Example: Probability of the Complement of an Event

You survey a sample of 1000 employees at a company and record the age of each. Find the probability of randomly choosing an employee who is not between 25 and 34 years old.

<table>
<thead>
<tr>
<th>Employee ages</th>
<th>Frequency, $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 to 24</td>
<td>54</td>
</tr>
<tr>
<td>25 to 34</td>
<td>366</td>
</tr>
<tr>
<td>35 to 44</td>
<td>233</td>
</tr>
<tr>
<td>45 to 54</td>
<td>180</td>
</tr>
<tr>
<td>55 to 64</td>
<td>125</td>
</tr>
<tr>
<td>65 and over</td>
<td>42</td>
</tr>
<tr>
<td>$\Sigma f = 1000$</td>
<td></td>
</tr>
</tbody>
</table>
Solution: Probability of the Complement of an Event

• Use empirical probability to find \( P(\text{age 25 to 34}) \)

\[
P(\text{age 25 to 34}) = \frac{f}{n} = \frac{366}{1000} = 0.366
\]

• Use the complement rule

\[
P(\text{age is not 25 to 34}) = 1 - \frac{366}{1000} = \frac{634}{1000} = 0.634
\]

<table>
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</tr>
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<td>42</td>
</tr>
<tr>
<td>( \Sigma f = 1000 )</td>
<td></td>
</tr>
</tbody>
</table>
Example: Probability Using a Tree Diagram

A probability experiment consists of tossing a coin and spinning the spinner shown. The spinner is equally likely to land on each number. Use a tree diagram to find the probability of tossing a tail and spinning an odd number.
Solution: Probability Using a Tree Diagram

Tree Diagram:

\[
P(\text{tossing a tail and spinning an odd number}) = \frac{4}{16} = \frac{1}{4} = 0.25
\]
Example: Probability Using the Fundamental Counting Principle

Your college identification number consists of 8 digits. Each digit can be 0 through 9 and each digit can be repeated. What is the probability of getting your college identification number when randomly generating eight digits?
Solution: Probability Using the Fundamental Counting Principle

• Each digit can be repeated
• There are 10 choices for each of the 8 digits
• Using the Fundamental Counting Principle, there are
  \[10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^8 = 100,000,000\] possible identification numbers
• Only one of those numbers corresponds to your ID number

\[P(\text{your ID number}) = \frac{1}{100,000,000}\]
Conditional Probability

Conditional Probability

• The probability of an event occurring, given that another event has already occurred
• Denoted $P(B \mid A)$ (read “probability of $B$, given $A$’”)
Example: Finding Conditional Probabilities

Two cards are selected in sequence from a standard deck. Find the probability that the second card is a queen, given that the first card is a king. (Assume that the king is not replaced.)

Solution:
Because the first card is a king and is not replaced, the remaining deck has 51 cards, 4 of which are queens.

\[
P(B \mid A) = P(2^{nd} \text{ card is a Queen} \mid 1^{st} \text{ card is a King}) = \frac{4}{51} \approx 0.078
\]
Example: Finding Conditional Probabilities

The table shows the results of a study in which researchers examined a child’s IQ and the presence of a specific gene in the child. Find the probability that a child has a high IQ, given that the child has the gene.

<table>
<thead>
<tr>
<th></th>
<th>Gene Present</th>
<th>Gene not present</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>High IQ</td>
<td>33</td>
<td>19</td>
<td>52</td>
</tr>
<tr>
<td>Normal IQ</td>
<td>39</td>
<td>11</td>
<td>50</td>
</tr>
<tr>
<td>Total</td>
<td>72</td>
<td>30</td>
<td>102</td>
</tr>
</tbody>
</table>
Solution: Finding Conditional Probabilities

There are 72 children who have the gene. So, the sample space consists of these 72 children.

<table>
<thead>
<tr>
<th></th>
<th>Gene Present</th>
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<td>72</td>
<td>30</td>
<td>102</td>
</tr>
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</table>

Of these, 33 have a high IQ.

\[ P(B \mid A) = P(\text{high IQ} \mid \text{gene present}) = \frac{33}{72} \approx 0.458 \]
Independent and Dependent Events

Independent events

- The occurrence of one of the events does not affect the probability of the occurrence of the other event
- \( P(B \mid A) = P(B) \) or \( P(A \mid B) = P(A) \)
- Events that are not independent are dependent
Example: Independent and Dependent Events

Decide whether the events are independent or dependent.

1. Selecting a king from a standard deck ($A$), not replacing it, and then selecting a queen from the deck ($B$).

Solution:

\[
P(B \mid A) = P(2^{nd} \text{ card is a Queen} \mid 1^{st} \text{ card is a King}) = \frac{4}{51}
\]

\[
P(B) = P(\text{Queen}) = \frac{4}{52}
\]

Dependent (the occurrence of $A$ changes the probability of the occurrence of $B$)
Example: Independent and Dependent Events

Decide whether the events are independent or dependent.

2. Tossing a coin and getting a head (A), and then rolling a six-sided die and obtaining a 6 (B).

Solution:

\[ P(B \mid A) = P(\text{rolling a 6} \mid \text{head on coin}) = \frac{1}{6} \]

\[ P(B) = P(\text{rolling a 6}) = \frac{1}{6} \]

Independent (the occurrence of A does not change the probability of the occurrence of B)
The Multiplication Rule

Multiplication rule for the probability of $A$ and $B$

- The probability that two events $A$ and $B$ will occur in sequence is
  \[ P(A \text{ and } B) = P(A) \cdot P(B \mid A) \]
- For independent events the rule can be simplified to
  \[ P(A \text{ and } B) = P(A) \cdot P(B) \]
- Can be extended for any number of independent events
Example: Using the Multiplication Rule

Two cards are selected, without replacing the first card, from a standard deck. Find the probability of selecting a king and then selecting a queen.

Solution:
Because the first card is not replaced, the events are dependent.

\[ P(K \text{ and } Q) = P(K) \cdot P(Q \mid K) \]
\[ = \frac{4}{52} \cdot \frac{4}{51} \]
\[ = \frac{16}{2652} \approx 0.006 \]
Example: Using the Multiplication Rule

A coin is tossed and a die is rolled. Find the probability of getting a head and then rolling a 6.

Solution:
The outcome of the coin does not affect the probability of rolling a 6 on the die. These two events are independent.

\[ P(H \text{ and } 6) = P(H) \cdot P(6) \]

\[ = \frac{1}{2} \cdot \frac{1}{6} \]

\[ = \frac{1}{12} \approx 0.083 \]
Example: Using the Multiplication Rule

The probability that a particular knee surgery is successful is 0.85. Find the probability that three knee surgeries are successful.

Solution:
The probability that each knee surgery is successful is 0.85. The chance for success for one surgery is independent of the chances for the other surgeries.

\[ P(3 \text{ surgeries are successful}) = (0.85)(0.85)(0.85) \approx 0.614 \]
Example: Using the Multiplication Rule

Find the probability that none of the three knee surgeries is successful.

Solution:
Because the probability of success for one surgery is 0.85. The probability of failure for one surgery is $1 - 0.85 = 0.15$

$$P(\text{none of the 3 surgeries is successful}) = (0.15)(0.15)(0.15) \approx 0.003$$
Example: Using the Multiplication Rule

Find the probability that at least one of the three knee surgeries is successful.

Solution:

“At least one” means one or more. The complement to the event “at least one successful” is the event “none are successful.” Using the complement rule

\[
P(\text{at least 1 is successful}) = 1 - P(\text{none are successful}) \\ \approx 1 - 0.003 \\ = 0.997
\]
Example: Using the Multiplication Rule to Find Probabilities

More than 15,000 U.S. medical school seniors applied to residency programs in 2009. Of those, 93% were matched to a residency position. Eighty-two percent of the seniors matched to a residency position were matched to one of their top two choices. Medical students electronically rank the residency programs in their order of preference and program directors across the United States do the same. The term “match” refers to the process where a student’s preference list and a program director’s preference list overlap, resulting in the placement of the student for a residency position. *(Source: National Resident Matching Program)*

(continued)
Example: Using the Multiplication Rule to Find Probabilities

1. Find the probability that a randomly selected senior was matched a residency position \( \text{and} \) it was one of the senior’s top two choices.

Solution:

\[ A = \{ \text{matched to residency position} \} \]
\[ B = \{ \text{matched to one of two top choices} \} \]

\[ P(A) = 0.93 \quad \text{and} \quad P(B \mid A) = 0.82 \]

\[ P(A \text{ and } B) = P(A) \cdot P(B \mid A) = (0.93)(0.82) \approx 0.763 \]

dependent events
Example: Using the Multiplication Rule to Find Probabilities

2. Find the probability that a randomly selected senior that was matched to a residency position did not get matched with one of the senior’s top two choices.

Solution:
Use the complement:

\[ P(B' \mid A) = 1 - P(B \mid A) \]

\[ = 1 - 0.82 = 0.18 \]
Mutually Exclusive Events

Mutually exclusive

- Two events $A$ and $B$ cannot occur at the same time

$A$ and $B$ are mutually exclusive

$A$ and $B$ are not mutually exclusive
Example: Mutually Exclusive Events

Decide if the events are mutually exclusive.

Event $A$: Roll a 3 on a die.
Event $B$: Roll a 4 on a die.

Solution:
Mutually exclusive (The first event has one outcome, a 3. The second event also has one outcome, a 4. These outcomes cannot occur at the same time.)
Example: Mutually Exclusive Events

Decide if the events are mutually exclusive.

Event $A$: Randomly select a male student.

Event $B$: Randomly select a nursing major.

Solution:
Not mutually exclusive (The student can be a male nursing major.)
The Addition Rule

Addition rule for the probability of $A$ or $B$

- The probability that events $A$ or $B$ will occur is
  \[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]
- For mutually exclusive events $A$ and $B$, the rule can be simplified to
  \[ P(A \text{ or } B) = P(A) + P(B) \]
- Can be extended to any number of mutually exclusive events
Example: Using the Addition Rule

You select a card from a standard deck. Find the probability that the card is a 4 or an ace.

Solution:
The events are mutually exclusive (if the card is a 4, it cannot be an ace)

\[
P(4 \text{ or ace}) = P(4) + P(\text{ace})
\]

\[
= \frac{4}{52} + \frac{4}{52}
\]

\[
= \frac{8}{52} \approx 0.154
\]
Example: Using the Addition Rule

You roll a die. Find the probability of rolling a number less than 3 or rolling an odd number.

Solution:
The events are not mutually exclusive (1 is an outcome of both events)
Solution: Using the Addition Rule

Roll a Die

\[
P(\text{less than 3 or odd})
\]

\[
= P(\text{less than 3}) + P(\text{odd}) - P(\text{less than 3 and odd})
\]

\[
= \frac{2}{6} + \frac{3}{6} - \frac{1}{6} = \frac{4}{6} \approx 0.667
\]
Example: Using the Addition Rule

The frequency distribution shows the volume of sales (in dollars) and the number of months a sales representative reached each sales level during the past three years. If this sales pattern continues, what is the probability that the sales representative will sell between $75,000 and $124,999 next month?

<table>
<thead>
<tr>
<th>Sales volume ($)</th>
<th>Months</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–24,999</td>
<td>3</td>
</tr>
<tr>
<td>25,000–49,999</td>
<td>5</td>
</tr>
<tr>
<td>50,000–74,999</td>
<td>6</td>
</tr>
<tr>
<td>75,000–99,999</td>
<td>7</td>
</tr>
<tr>
<td>100,000–124,999</td>
<td>9</td>
</tr>
<tr>
<td>125,000–149,999</td>
<td>2</td>
</tr>
<tr>
<td>150,000–174,999</td>
<td>3</td>
</tr>
<tr>
<td>175,000–199,999</td>
<td>1</td>
</tr>
</tbody>
</table>
Solution: Using the Addition Rule

- \( A = \) monthly sales between $75,000 and $99,999
- \( B = \) monthly sales between $100,000 and $124,999
- \( A \) and \( B \) are mutually exclusive

\[
P(A \text{ or } B) = P(A) + P(B)
\]
\[
= \frac{7}{36} + \frac{9}{36}
\]
\[
= \frac{16}{36} \approx 0.444
\]
Example: Using the Addition Rule

A blood bank catalogs the types of blood given by donors during the last five days. A donor is selected at random. Find the probability the donor has type O or type A blood.

<table>
<thead>
<tr>
<th>Type</th>
<th>Rh-Positive</th>
<th>Rh-Negative</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type O</td>
<td>156</td>
<td>28</td>
<td>184</td>
</tr>
<tr>
<td>Type A</td>
<td>139</td>
<td>25</td>
<td>164</td>
</tr>
<tr>
<td>Type B</td>
<td>37</td>
<td>8</td>
<td>45</td>
</tr>
<tr>
<td>Type AB</td>
<td>12</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>Total</td>
<td>344</td>
<td>65</td>
<td>409</td>
</tr>
</tbody>
</table>
Solution: Using the Addition Rule

The events are mutually exclusive (a donor cannot have type O blood and type A blood)

<table>
<thead>
<tr>
<th></th>
<th>Type O</th>
<th>Type A</th>
<th>Type B</th>
<th>Type AB</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rh-Positive</td>
<td>156</td>
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<td>28</td>
<td>25</td>
<td>8</td>
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<td>184</td>
<td>164</td>
<td>45</td>
<td>16</td>
<td>409</td>
</tr>
</tbody>
</table>

\[ P(\text{type O or type A}) = P(\text{type O}) + P(\text{type A}) \]

\[ = \frac{184}{409} + \frac{164}{409} \]

\[ = \frac{348}{409} \approx 0.851 \]
Example: Using the Addition Rule

Find the probability the donor has type B or is Rh-negative.

<table>
<thead>
<tr>
<th></th>
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<th>Type A</th>
<th>Type B</th>
<th>Type AB</th>
<th>Total</th>
</tr>
</thead>
<tbody>
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<td>Rh-Positive</td>
<td>156</td>
<td>139</td>
<td>37</td>
<td>12</td>
<td>344</td>
</tr>
<tr>
<td>Rh-Negative</td>
<td>28</td>
<td>25</td>
<td>8</td>
<td>4</td>
<td>65</td>
</tr>
<tr>
<td>Total</td>
<td>184</td>
<td>164</td>
<td>45</td>
<td>16</td>
<td>409</td>
</tr>
</tbody>
</table>

Solution:
The events are not mutually exclusive (a donor can have type B blood and be Rh-negative)
Solution: Using the Addition Rule

<table>
<thead>
<tr>
<th></th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rh-Positive</strong></td>
<td>344</td>
</tr>
<tr>
<td><strong>Rh-Negative</strong></td>
<td>65</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>409</td>
</tr>
</tbody>
</table>

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<td>37</td>
<td>12</td>
<td>344</td>
</tr>
<tr>
<td><strong>Rh-Negative</strong></td>
<td>28</td>
<td>25</td>
<td>8</td>
<td>4</td>
<td>65</td>
</tr>
</tbody>
</table>

\[
P(\text{type B or Rh\text – neg}) = P(\text{type B}) + P(\text{Rh\text – neg}) - P(\text{type B and Rh\text – neg})
\]

\[
= \frac{45}{409} + \frac{65}{409} - \frac{8}{409} = \frac{102}{409} \approx 0.249
\]
Permutations

Permutation

- An ordered arrangement of objects
- The number of different permutations of \( n \) distinct objects is \( n! \) (\( n \) factorial)
  - \( n! = n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \)
  - \( 0! = 1 \)
- Examples:
  - \( 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \)
  - \( 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \)
Example: Permutation of $n$ Objects

The objective of a $9 \times 9$ Sudoku number puzzle is to fill the grid so that each row, each column, and each $3 \times 3$ grid contain the digits 1 to 9. How many different ways can the first row of a blank $9 \times 9$ Sudoku grid be filled?

Solution:
The number of permutations is

$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362,880 \text{ ways}$$
Permutations

Permutation of $n$ objects taken $r$ at a time

- The number of different permutations of $n$ distinct objects taken $r$ at a time

\[ n \, P_r = \frac{n!}{(n - r)!}, \text{ where } r \leq n \]
Example: Finding $nPr$

Find the number of ways of forming four-digit codes in which no digit is repeated.

**Solution:**

- You need to select 4 digits from a group of 10
- $n = 10$, $r = 4$

\[
10 \, P_4 = \frac{10!}{(10 - 4)!} = \frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 5040 \text{ ways}
\]
Example: Finding $nPr$

Forty-three race cars started the 2007 Daytona 500. How many ways can the cars finish first, second, and third?

Solution:
• You need to select 3 cars from a group of 43
• $n = 43, \ r = 3$

$$\binom{43}{3} P_3 = \frac{43!}{(43-3)!} = \frac{43!}{40!}$$

$$= 43 \cdot 42 \cdot 41$$

$$= 74,046 \text{ ways}$$
Distinguishable Permutations

• The number of distinguishable permutations of \( n \) objects where \( n_1 \) are of one type, \( n_2 \) are of another type, and so on

\[
\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_k!}
\]

where \( n_1 + n_2 + n_3 + \cdots + n_k = n \)
Example: Distinguishable Permutations

A building contractor is planning to develop a subdivision that consists of 6 one-story houses, 4 two-story houses, and 2 split-level houses. In how many distinguishable ways can the houses be arranged?

Solution:

• There are 12 houses in the subdivision
• $n = 12, \ n_1 = 6, \ n_2 = 4, \ n_3 = 2$

\[
\frac{12!}{6! \cdot 4! \cdot 2!} = 13,860 \text{ distinguishable ways}
\]
Combinations

Combination of $n$ objects taken $r$ at a time

- A selection of $r$ objects from a group of $n$ objects without regard to order

\[
\binom{n}{r} = \frac{n!}{(n-r)!r!}, \quad \text{where } r \leq n
\]
Example: Combinations

A state’s department of transportation plans to develop a new section of interstate highway and receives 16 bids for the project. The state plans to hire four of the bidding companies. How many different combinations of four companies can be selected from the 16 bidding companies?

Solution:

- You need to select 4 companies from a group of 16
- \( n = 16, \ r = 4 \)
- Order is not important
Solution: Combinations

\[ \binom{16}{4} = \frac{16!}{(16-4)!4!} = \frac{16!}{12!4!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{12! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1820 \text{ different combinations} \]
Example: Finding Probabilities

A student advisory board consists of 17 members. Three members serve as the board’s chair, secretary, and webmaster. Each member is equally likely to serve any of the positions. What is the probability of selecting at random the three members that hold each position?
Solution: Finding Probabilities

- There is only one favorable outcome.
- There are \( \binom{17}{3} \) ways the three positions can be filled.

\[
\binom{17}{3} = \frac{17!}{(17-3)!} = \frac{17!}{14!} = 17 \cdot 16 \cdot 15 = 4080
\]

ways the three positions can be filled.

\[
P(\text{selecting the 3 members}) = \frac{1}{4080} \approx 0.0002
\]
Example: Finding Probabilities

You have 11 letters consisting of one M, four Is, four Ss, and two Ps. If the letters are randomly arranged in order, what is the probability that the arrangement spells the word *Mississippi*?
Solution: Finding Probabilities

- There is only one favorable outcome
- There are \( \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34,650 \) distinguishable permutations of the given letters

\[ P(\text{Mississippi}) = \frac{1}{34650} \approx 0.000029 \]
Example: Finding Probabilities

A food manufacturer is analyzing a sample of 400 corn kernels for the presence of a toxin. In this sample, three kernels have dangerously high levels of the toxin. If four kernels are randomly selected from the sample, what is the probability that exactly one kernel contains a dangerously high level of the toxin?
Solution: Finding Probabilities

• The possible number of ways of choosing one toxic kernel out of three toxic kernels is
  \[ _3C_1 = 3 \]

• The possible number of ways of choosing three nontoxic kernels from 397 nontoxic kernels is
  \[ _{397}C_3 = 10,349,790 \]

• Using the Multiplication Rule, the number of ways of choosing one toxic kernel and three nontoxic kernels is
  \[ _3C_1 \cdot _{397}C_3 = 3 \cdot 10,349,790 = 31,049,370 \]
Solution: Finding Probabilities

- The number of possible ways of choosing 4 kernels from 400 kernels is
  \[ \binom{400}{4} = 1,050,739,900 \]

- The probability of selecting exactly 1 toxic kernel is
  \[ P(1 \text{ toxic kernel}) = \frac{\binom{3}{1} \cdot \binom{397}{3}}{\binom{400}{4}} \approx \frac{31,049,370}{1,050,739,900} \approx 0.0296 \]